Beam Design and Deflections

Notation:

\( a \) = name for width dimension  
\( A \) = name for area  
\( A_{\text{req'd}-\text{adj}} \) = area required at allowable stress when shear is adjusted to include self weight  
\( A_{\text{web}} \) = area of the web of a wide flange section  
\( b \) = width of a rectangle  
\( b \) = total width of material at a horizontal section  
\( b \) = name for height dimension  
\( c \) = largest distance from the neutral axis to the top or bottom edge of a beam  
\( c_1 \) = coefficient for shear stress for a rectangular bar in torsion  
\( d \) = calculus symbol for differentiation  
\( DL \) = shorthand for dead load  
\( E \) = modulus of elasticity  
\( f_b \) = bending stress  
\( f_p \) = bearing stress (see \( P \))  
\( f_v \) = shear stress  
\( f_{v-\text{max}} \) = maximum shear stress  
\( F_b \) = allowable bending stress  
\( F_v \) = allowable shear stress  
\( F_p \) = allowable bearing stress  
\( F_y \) = yield strength  
\( F_{y\text{web}} \) = yield strength of the web material  
\( h \) = height of a rectangle  
\( I \) = moment of inertia with respect to neutral axis bending  
\( I_{\text{trial}} \) = moment of inertia of trial section  
\( I_{\text{req'd}} \) = moment of inertia required at limiting deflection  
\( J \) = polar moment of inertia  
\( L \) = name for span length  
\( LL \) = shorthand for live load  
\( LRFD \) = load and resistance factor design  
\( M \) = internal bending moment  
\( M_{\text{max}} \) = maximum internal bending moment  
\( M_{\text{max}-\text{adj}} \) = maximum bending moment adjusted to include self weight  
\( M_n \) = nominal flexure strength with the full section at the yield stress for LRFD  
\( M_u \) = maximum moment from factored loads for LRFD  
\( P \) = name for axial force vector  
\( Q \) = first moment area about a neutral axis  
\( R \) = radius of curvature of a deformed beam  
\( S \) = section modulus  
\( S_{\text{req'd}} \) = section modulus required at allowable stress  
\( T \) = torque (axial moment)  
\( V \) = internal shear force  
\( V_{\text{max}} \) = maximum internal shear force  
\( V_{\text{max}-\text{adj}} \) = maximum internal shear force adjusted to include self weight  
\( V_u \) = maximum shear from factored loads for LRFD  
\( w \) = name for distributed load  
\( w_{\text{self wt}} \) = name for distributed load from self weight of member  
\( x \) = horizontal distance  
\( y \) = vertical distance  
\( \Delta_{\text{actual}} \) = actual beam deflection  
\( \Delta_{\text{allowable}} \) = allowable beam deflection  
\( \Delta_{\text{limit}} \) = allowable beam deflection limit  
\( \Delta_{\text{max}} \) = maximum beam deflection  
\( \phi_b \) = resistance factor for flexure in LRFD design  
\( \phi_v \) = resistance factor for shear for LRFD  
\( \gamma \) = density or unit weight  
\( \theta \) = slope of the beam deflection curve  
\( \rho \) = radial distance  
\( \int \) = symbol for integration  
\( \Sigma \) = summation symbol
Criteria for Design

Allowable bending stress or bending stress from LRFD should not be exceeded:

\[ F_b \geq f_b = \frac{Mc}{I} \]

Knowing M and \( F_b \), the minimum section modulus fitting the limit is:

\[ S_{req'd} \geq \frac{M}{F_b} \]

Besides strength, we also need to be concerned about serviceability. This involves things like limiting deflections & cracking, controlling noise and vibrations, preventing excessive settlements of foundations and durability. When we know about a beam section and its material, we can determine beam deformations.

Determining Maximum Bending Moment

Drawing V and M diagrams will show us the maximum values for design. Remember:

\[ V = \Sigma(-w)dx \]
\[ M = \Sigma(V)dx \]
\[ \frac{dV}{dx} = -w \]
\[ \frac{dM}{dx} = V \]

Determining Maximum Bending Stress

For a prismatic member (constant cross section), the maximum normal stress will occur at the maximum moment.

For a non-prismatic member, the stress varies with the cross section AND the moment.

Deflections

If the bending moment changes, M(x) across a beam of constant material and cross section then the curvature will change:

\[ \frac{1}{R} = \frac{M(x)}{EI} \]

The slope of the n.a. of a beam, \( \theta \), will be tangent to the radius of curvature, R:

\[ \theta = \text{slope} = \frac{1}{EI} \int M(x)dx \]

The equation for deflection, y, along a beam is:

\[ y = \frac{1}{EI} \int \theta dx = \frac{1}{EI} \int \int M(x)dx \]

Elastic curve equations can be found in handbooks, textbooks, design manuals, etc...Computer programs can be used as well.

Elastic curve equations can be superpositioned ONLY if the stresses are in the elastic range.
The deflected shape is roughly the same shape as the bending moment diagram flipped but is constrained by supports and geometry.

Boundary Conditions

The boundary conditions are geometrical values that we know – slope or deflection – which may be restrained by supports or symmetry.

At Pins, Rollers, Fixed Supports: \( y = 0 \)

At Fixed Supports: \( \theta = 0 \)

At Inflection Points From Symmetry: \( \theta = 0 \)

The Slope Is Zero At The Maximum Deflection \( y_{\text{max}} \):

\[
\theta = \frac{dy}{dx} = \text{slope} = 0
\]

Allowable Deflection Limits

All building codes and design codes limit deflection for beam types and damage that could happen based on service condition and severity.

\[
y_{\text{max}}(x) = \Delta_{\text{actual}} \leq \Delta_{\text{allowable}} = \frac{L}{\text{value}}
\]

<table>
<thead>
<tr>
<th>Use</th>
<th>LL only</th>
<th>DL+LL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roof beams:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Industrial (no ceiling)</td>
<td>L/180</td>
<td>L/120</td>
</tr>
<tr>
<td>Commercial</td>
<td></td>
<td></td>
</tr>
<tr>
<td>plaster ceiling</td>
<td>L/240</td>
<td>L/180</td>
</tr>
<tr>
<td>no plaster</td>
<td>L/360</td>
<td>L/240</td>
</tr>
<tr>
<td>Floor beams:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ordinary Usage</td>
<td>L/360</td>
<td>L/240</td>
</tr>
<tr>
<td>Roof or floor (damageable elements)</td>
<td>L/480</td>
<td></td>
</tr>
</tbody>
</table>
Beam Loads & Load Tracing

In order to determine the loads on a beam (or girder, joist, column, frame, foundation...) we can start at the top of a structure and determine the *tributary area* that a load acts over and the beam needs to support. Loads come from material weights, people, and the environment. This area is assumed to be from half the distance to the next beam over to halfway to the next beam.

The reactions must be supported by the next lower structural element *ad infinitum*, to the ground.

Design Procedure

The intent is to find the most light weight member satisfying the section modulus size.

1. Know \( F_b \) (allowable stress) for the material or \( F_y \) & \( F_u \) for LRFD.
2. Draw \( V \) & \( M \), finding \( M_{\text{max}} \).
3. Calculate \( S_{\text{req'd}} \). This step is equivalent to determining \( f_b = \frac{M_{\text{max}}}{S} \leq F_b \)
4. For rectangular beams \( S = \frac{bh^2}{6} \)
   - For steel or timber: use the section charts to find \( S \) that will work and remember that the beam self weight will increase \( S_{\text{req'd}} \). And for steel, the design charts show the lightest section within a grouping of similar \( S \)'s. \( w_{\text{self wt}} = \gamma A \)
   - For any thing else, try a nice value for \( b \), and calculate \( h \) or the other way around.

   ***Determine the “updated” \( V_{\text{max}} \) and \( M_{\text{max}} \) including the beam self weight, and verify that the updated \( S_{\text{req'd}} \) has been met.******

5. Consider lateral stability

6. Evaluate horizontal shear stresses using \( V_{\text{max}} \) to determine if \( f_v \leq F_v \)

   For rectangular beams, \( W \)'s, and others: \( f_{v-\text{max}} = \frac{3V}{2A} \approx \frac{V}{A_{\text{web}}} \) or \( \frac{VQ}{I_b} \)

7. Provide adequate bearing area at supports: \( f_p = \frac{P}{A} \leq F_p \)

8. Evaluate shear due to torsion \( f_v = \frac{T_p}{J} \) or \( \frac{T}{c_fab^2} \leq F_v \)

   (circular section or rectangular)

9. Evaluate the deflection to determine if \( \Delta_{\text{max LL}} \leq \Delta_{\text{LL-allowed}} \) and/or \( \Delta_{\text{maxTotal}} \leq \Delta_{\text{T-allowed}} \)

   ***note: when \( \Delta_{\text{calculated}} > \Delta_{\text{limit}} \), \( I_{\text{required}} \) can be found with: and \( S_{\text{req'd}} \) will be satisfied for similar self weight ****** \( I_{\text{req'd}} \geq \frac{\Delta_{\text{no big}}}{\Delta_{\text{limit}}} I_{\text{trial}} \)

FOR ANY EVALUATION:

Redesign (with a new section) at any point that a stress or serviceability criteria is NOT satisfied and re-evaluate each condition until it is satisfactory.
Example 1 (changed from pg 284) (superpositioning)
Example Problem 8.5 (Semi-Graphical Method)

A cantilever beam supports a uniform load of $\omega = 2 \text{kN/m}$ over its entire span, plus a concentrated load of 10 kN at the free end. Investigate using Beam Diagrams and Formulas.

SOLUTION:

By examining the support conditions, we are looking for a cantilevered beam from Cases 18 through 23.

There is a case for uniformly distributed load across the span (19) and for a load at any point (21).

For both these cases, it shows that the maximum shear AND maximum moment are located at the fixed end. If we add the values, the shear and diagrams should look like this:

We can find the maximum shear (at B) from

$$V = P + \omega l = 10 \text{kN} + 2 \text{kN/m} \times 3 \text{m} = 16 \text{kN}$$

The maximum moment (at B) will be

$$M = Pb + \omega l^2/2 = 10 \text{kN} \times 2.25 \text{m} + 2 \text{kN/m} (3 \text{m})^2/2 = 31.5 \text{kN-m}$$

The key values for the diagrams can be found with the general equations ($V_x$ and $M_x$):

$$V_C^- = -\omega x = -2 \text{kN/m} \times 0.25 \text{m} = -0.5 \text{kN}; V_C^+ = -P - \omega x = -10 \text{kN} - 2 \text{kN/m} \times 0.25 \text{m} = -10.5 \text{kN}$$

$$M_C = -\omega x^2/2 = 2 \text{kN/m} (0.25 \text{m})^2/2 = 0.0625 \text{kN-m}$$

We can find the maximum deflection by looking at the cases. Both say $\Delta_{\text{max}}$ (at free end), so the values can be added directly. Superpositioning of values must be at the same $x$ location. Assume $E = 70 \times 10^3$ MPa and $I = 45 \times 10^6$ mm$^4$

$$\Delta_{\text{end}} = \frac{Pb^2}{6EI} (3l - b) + \frac{\omega l^4}{8EI} = \frac{10'^6 (2.25m)^2 (10' / 1') (10' / 1')} {6(70 \times 10^3 \text{ MPa})(45 \times 10^6 \text{ mm}^4)} (3 \cdot 3m - 2.25m) + \frac{2'^6 (3m)^2 (10' / 1') (10' / 1')} {8(70 \times 10^3 \text{ MPa})(45 \times 10^6 \text{ mm}^4)}$$

$$= 18.08 \text{ mm} + 6.43 \text{ mm} = 24.5 \text{ mm}$$

21. CANTILEVER BEAM—CONCENTRATED LOAD AT ANY POINT

<table>
<thead>
<tr>
<th>Total Eqiv. Uniform Load</th>
<th>$= \frac{3Pb}{l}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = V$</td>
<td>$= P$</td>
</tr>
<tr>
<td>$M_{\text{max}}$ (at fixed end)</td>
<td>$= Pb$</td>
</tr>
<tr>
<td>$M_x$ (when $x &gt; a$)</td>
<td>$= P (x-a)$</td>
</tr>
<tr>
<td>$\Delta_{\text{max}}$ (at free end)</td>
<td>$= \frac{Pb^2 (3l-b)}{6EI}$</td>
</tr>
<tr>
<td>$\Delta_a$ (at point of load)</td>
<td>$= \frac{Pb^2 (3l-b)}{24EI}$</td>
</tr>
<tr>
<td>$\Delta_x$ (when $x &lt; a$)</td>
<td>$= \frac{Pb^2 (3l-3x-b)}{6EI}$</td>
</tr>
<tr>
<td>$\Delta_x$ (when $x &gt; a$)</td>
<td>$= \frac{P (x-a)^3 (3b-l+x)}{24EI}$</td>
</tr>
</tbody>
</table>

19. CANTILEVER BEAM—UNIFORMLY DISTRIBUTED LOAD

<table>
<thead>
<tr>
<th>Total Eqiv. Uniform Load</th>
<th>$= 4\omega l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = V$</td>
<td>$= \omega l$</td>
</tr>
<tr>
<td>$V_x$</td>
<td>$= \omega x$</td>
</tr>
<tr>
<td>$M_{\text{max}}$ (at fixed end)</td>
<td>$= \frac{\omega l^2}{2}$</td>
</tr>
<tr>
<td>$M_x$</td>
<td>$= \frac{\omega x^2}{2}$</td>
</tr>
<tr>
<td>$\Delta_{\text{max}}$ (at free end)</td>
<td>$= \frac{\omega l^4}{24EI}$</td>
</tr>
<tr>
<td>$\Delta_x$</td>
<td>$= \frac{\omega (x^4 - 4\omega x + 3)x}{24EI}$</td>
</tr>
</tbody>
</table>
Example 2 (pg 275) (superpositioning)
Example Problem 8.2 (Equilibrium Method)

Draw V and M diagrams for an overhang beam (Figure 8.12) loaded as shown. Determine the critical \( V_{\text{max}} \) and \( M_{\text{max}} \) locations and magnitudes using Beam Diagrams and Formulas.

SOLUTION:

By examining the support conditions, we are looking for beam with an overhang on one end from Cases 24 through 28. (Even though the overhang is on the right, and not on the left like our beam, we can still use the information by recognizing that we can mirror the figure about the left end.)

There is a case for a load at the end (26) but none for a load in between the supports. This is because it behaves exactly like a simply supported beam in this instance (no shear or bending on the overhang). The case for this is #5 (reversed again).

If we "flip" the diagrams (both vertically and horizontally) and add the values, the resulting shear and bending moment should look like this:

We still have to find the peak values of shear and the location of the zero shear to find the critical moment values.

(Notice \( R_1 \) is shown down.)

\[
R_{1(0)} = \frac{-Pa}{l} + \frac{wa}{2l} \left( 2l - a \right) = \frac{-10^4 \cdot 10^6}{20} + \frac{2 \cdot 10^3 \cdot 10^5}{2 \cdot 20} (2 \cdot 20^6 - 10^6) = 10^6
\]

\[
R_{2(0)} = 10^6 + 2 \cdot 10^3 \cdot 10^5 - 10^6 = 10^6
\]

(from the total downward load – \( R_{1(0)} \))

\[
V_A = -10^6 \quad V_B = \left( -10^6 + 20^6 \right) = 10^6
\]

\[
V_B = 10^6 - 2 \cdot 10^5 \text{(at } B\text{)} = 10^6 \quad M_B = -10^4 \text{(at } B\text{)}
\]

\[
M_C = -100 \text{ k-ft} + 10 \text{ k}(10 \text{ ft}) = 0
\]

\[
M_x = 0 + 10 \text{ k}(5 \text{ ft})/2 = 25 \text{ k-ft}
\]

We can calculate the deflection between the supports. (And at the end for case 5 if we derive the slope!)

Assume \( E = 29 \times 10^3 \text{ ksi and } I = 103 \text{ in}^4 \)

We'll investigate the maximum between the supports from case 26 (because it isn't obvious where the maximum will be.)

\[
x = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} = 11.55 \text{ ft} \quad \text{(to left of D)} \quad \text{and} \quad \Delta_{\text{max}} = \Delta_{\text{max, -over 26}} + \Delta_{\text{over 26}} \quad \text{with } x \text{ (11.55 ft) greater than } a \text{ (10 ft)}:
\]

\[
\Delta_{\text{max}} = \frac{0.6415 \cdot Pa^2}{EI} \quad \frac{wa}{24EI} \left( 4x^2 - 2x^3 - a^3 \right)
\]

\[
\Delta_{\text{max}} = -0.6415 \quad \frac{10^4 \left( 10^6 \right) \left( 12^{-3} \right) \left( 12^{-3} \right)}{24 \left( 103 \text{ in}^4 \right)} \times \frac{-2 \cdot 10^5 \left( 10^6 \right) \left( 20^6 - 10^6 \right)}{24 \left( 9 \times 10^9 \text{ ksi} \right) \left( 103 \text{ in}^4 \right) \left( 20^6 \right)}
\]

\[
(4 \cdot 11.55^2 \cdot 20^6 - 2(11.55^2) \cdot (10^6) \cdot 1(12^{-3} \cdot 12^{-3})
\]

\[
= -1.484 \text{ in} + 1.343 \text{ in} = -0.141 \text{ in (up)}
\]
Beam Design Flow Chart

1. Collect data: L, \( \alpha \), \( \gamma \), \( \Delta_{\text{limits}} \); find beam charts for load cases and \( \Delta_{\text{actual}} \) equations

2. ASD
   - Allowable Stress or LRFD Design?

3. Collect data: \( F_b \) & \( F_v \)

4. Find \( V_{\text{max}} \) & \( M_{\text{max}} \) from constructing diagrams or using beam chart formulas

5. Find \( S_{\text{req'd}} \) and pick a section from a table with \( S_x \) greater or equal to \( S_{\text{req'd}} \)

6. Calculate \( \alpha_{\text{eff, wt.}} \) using \( A \) found and \( \gamma \). Find \( M_{\text{max-adj}} \) & \( V_{\text{max-adj}} \)

7. No
   - Calculate \( S_{\text{req'd-adj}} \) using \( M_{\text{max-adj}} \)
     - Is \( S_{\text{picked}} \) \( \geq \) \( S_{\text{req'd-adj}} \)?
       - OR calculate \( f_b \) Is \( f_b \) \( \leq \) \( F_b \)?

8. Yes
   - Calculate \( A_{\text{req'd-adj}} \) using \( V_{\text{max-adj}} \)
     - Is \( A_{\text{picked}} \) \( \geq \) \( A_{\text{req'd-adj}} \)?
       - OR calculate \( f_v \) Is \( f_v \) \( \leq \) \( F_v \)?

9. No
   - Calculate \( \Delta_{\text{max}} \) (no load factors!) using superpositioning and beam chart equations with the \( I_x \) for the section

10. \( I_{\text{req'd}} \geq \frac{\Delta_{\text{max-req'd}}}{\Delta_{\text{max-trial}}} \)

11. Is \( \Delta_{\text{max}} \leq \Delta_{\text{limits}} \)?

   - Yes
     - This may be both the limit for live load deflection and total load deflection.
     - (DONE)

   - No
     - No pick a section with a larger \( I_x \)

12. No
    - Pick a new section with a larger area

13. Yes
    - Pick a section with a larger web area

14. Is \( V_u \leq \phi_v(0.6F_{\text{y, web}}A_{\text{web}}) \)?

15. No
    - pick a section with a larger web area

16. Yes
    - Is \( V_{\text{u}} \leq \phi_v(0.6F_{\text{y, web}}A_{\text{web}}) \)?