Centers of Gravity - Centroids

Notation:
- \( A \): name for area
- \( C \): designation for channel section
- \( F \): name for centroid
- \( F_z \): force component in the z direction
- \( L \): name for length
- \( O \): name for reference origin
- \( Q_x \): first moment area about an x axis (using y distances)
- \( Q_y \): first moment area about an y axis (using x distances)
- \( t \): name for thickness
- \( t_w \): thickness of web of wide flange
- \( W \): name for force due to weight
- \( W \): designation for wide flange section
- \( x \): horizontal distance
- \( \bar{x} \): the distance in the x direction from a reference axis to the centroid of a shape
- \( \hat{x} \): the distance in the x direction from a reference axis to the centroid of a composite shape
- \( y \): vertical distance
- \( \bar{y} \): the distance in the y direction from a reference axis to the centroid of a shape
- \( \hat{y} \): the distance in the y direction from a reference axis to the centroid of a composite shape
- \( z \): distance perpendicular to x-y plane
- \( \Delta \): symbol for integration
- \( \Delta \): calculus symbol for small quantity
- \( \gamma \): density of a material (unit weight)
- \( \Sigma \): summation symbol

- The cross section shape and how it resists bending and twisting is important to understanding beam and column behavior.

- The center of gravity is the location of the equivalent force representing the total weight of a body comprised of particles that each have a mass gravity acts upon.

Resultant force: Over a body of constant thickness in x and y

\[
\sum F_z = \sum_{i=1}^{n} \Delta W_i = W \quad W = \int dW
\]

Location: \( \bar{x}, \bar{y} \) is the equivalent location of the force \( W \) from all \( \Delta W_i \)'s over all x & y locations (with respect to the moment from each force) from:

\[
\sum M_x = \sum_{i=1}^{n} x_i \Delta W_i = \bar{x}W \quad \bar{x}W = \int xdW \Rightarrow \bar{x} = \frac{\int xdW}{W} \quad \text{OR} \quad \bar{x} = \frac{\Sigma(x\Delta W)}{W}
\]

\[
\sum M_y = \sum_{i=1}^{n} y_i \Delta W_i = \bar{y}W \quad \bar{y}W = \int ydW \Rightarrow \bar{y} = \frac{\int ydW}{W} \quad \text{OR} \quad \bar{y} = \frac{\Sigma(y\Delta W)}{W}
\]
• The **centroid of an area** is the average x and y locations of the area particles

For a discrete shape ($\Delta A_i$) of a uniform thickness and material, the weight can be defined as:

$$\Delta W_i = \gamma t \Delta A_i$$

where:

- $\gamma$ is weight per unit **volume** (= specific weight) with units of $N/m^3$ or $lb/ft^3$
- $t\Delta A_i$ is the volume

So if $W = \gamma A$:

$$\bar{x}_A \gamma A = \int x \gamma dA \Rightarrow \bar{x}A = \int x dA \text{ OR } \bar{x} = \frac{\sum(x\Delta A)}{A} \text{ and similarly } \bar{y} = \frac{\sum(y\Delta A)}{A}$$

Similarly, for a line with constant cross section, $a$ ($\Delta W_i = \gamma a \Delta L_i$):

$$\bar{x}L = \int xdL \text{ OR } \bar{x} = \frac{\sum(x\Delta L)}{L} \text{ and } \bar{y}L = \int ydL \text{ OR } \bar{y} = \frac{\sum(y\Delta L)}{L}$$

- $\bar{x}$, $\bar{y}$ with respect to an x, y coordinate system is the centroid of an area AND the center of **gravity** for a body of uniform material and thickness.

• The **first moment of the area** is like a force moment: and is the area multiplied by the perpendicular distance to an axis.

$$Q_x = \int y dA = \bar{y}A \quad Q_y = \int x dA = \bar{x}A$$
Centroids of Common Shapes

Centroids of Common Shapes of Areas and Lines

<table>
<thead>
<tr>
<th>Shape</th>
<th>( \bar{x} )</th>
<th>( \bar{y} )</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular area</td>
<td>( \frac{b}{3} )</td>
<td>( \frac{h}{3} )</td>
<td>( \frac{bh}{2} )</td>
</tr>
<tr>
<td>Quarter-circular area</td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{\pi r^2}{4} )</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{\pi r^2}{2} )</td>
</tr>
<tr>
<td>Semiparabolic area</td>
<td>( \frac{3a}{8} )</td>
<td>( \frac{3h}{5} )</td>
<td>( \frac{2ah}{3} )</td>
</tr>
<tr>
<td>Parabolic area</td>
<td>0</td>
<td>( \frac{3h}{5} )</td>
<td>( \frac{4ah}{3} )</td>
</tr>
<tr>
<td>Parabolic span-drel</td>
<td>( \frac{3a}{4} )</td>
<td>( \frac{3h}{10} )</td>
<td>( \frac{ah}{3} )</td>
</tr>
<tr>
<td>Circular sector</td>
<td>( \frac{2r \sin \alpha}{3\alpha} )</td>
<td>0</td>
<td>( ar^2 )</td>
</tr>
<tr>
<td>Quarter-circular arc</td>
<td>( \frac{2r}{\pi} )</td>
<td>( \frac{2r}{\pi} )</td>
<td>( \frac{\pi r}{2} )</td>
</tr>
<tr>
<td>Semicircular arc</td>
<td>0</td>
<td>( \frac{2r}{\pi} )</td>
<td>( \pi r )</td>
</tr>
<tr>
<td>Arc of circle</td>
<td>( \frac{r \sin \alpha}{\alpha} )</td>
<td>0</td>
<td>( 2ar )</td>
</tr>
</tbody>
</table>
• Symmetric Areas

- An area is symmetric with respect to a line when every point on one side is mirrored on the other. The line divides the area into equal parts and the centroid will be on that axis.

- An area can be symmetric to a center point when every \((x,y)\) point is matched by a \((-x,-y)\) point. It does not necessarily have an axis of symmetry. The center point is the centroid.

- If the symmetry line is on an axis, the centroid location is on that axis (value of 0). With double symmetry, the centroid is at the intersection.

- Symmetry can also be defined by areas that match across a line, but are \(180^\circ\) to each other.

Basic Steps

1. Draw a reference origin.

2. Divide the area into basic shapes

3. Label the basic shapes (components)

4. Draw a table with headers of \(\text{Component}, \text{Area}, \bar{x}, \bar{x}A, \bar{y}, \bar{y}A\)

5. Fill in the table value

6. Draw a summation line. Sum all the areas, all the \(\bar{x}A\) terms, and all the \(\bar{y}A\) terms

7. Calculate \(\hat{x}\) and \(\hat{y}\)

• Composite Shapes

If we have a shape made up of basic shapes that we know centroid locations for, we can find an “average” centroid of the areas.

\[
\hat{x}A = \hat{x} \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \bar{x}_i A_i \\
\hat{y}A = \hat{y} \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \bar{y}_i A_i
\]

OR

\[
\hat{x} = \frac{\sum \bar{x} A}{A} \quad \hat{y} = \frac{\sum \bar{y} A}{A}
\]

Centroid values can be negative.

Area values can be negative (holes)
Example 1 (pg 243)

Example Problem 7.1: Centroids (Figures 7.5 and 7.6)

Determine the centroidal $x$ and $y$ distances for the composite area shown. Use the lower left corner of the trapezoid as the reference origin.

<table>
<thead>
<tr>
<th>Component</th>
<th>Area ($\Delta A$) (in.$^2$)</th>
<th>$\bar{x}$ (in.)</th>
<th>$\bar{x}\Delta A$ (in.$^3$)</th>
<th>$\bar{y}$ (in.)</th>
<th>$\bar{y}\Delta A$ (in.$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$\frac{9'(3')}{2} = 13.5$ in.$^2$</td>
<td>$6'$</td>
<td>$81$ in.$^3$</td>
<td>$4'$</td>
<td>$54$ in.$^3$</td>
</tr>
<tr>
<td>(b)</td>
<td>$9'(3') = 27$ in.$^2$</td>
<td>$4.5'$</td>
<td>$121.5$ in.$^3$</td>
<td>$1.5'$</td>
<td>$40.5$ in.$^3$</td>
</tr>
<tr>
<td></td>
<td>$A = \sum \Delta A = 40.5$ in.$^2$</td>
<td></td>
<td>$\sum \bar{x}\Delta A = 202.5$ in.$^3$</td>
<td></td>
<td>$\sum \bar{y}\Delta A = 94.5$ in.$^3$</td>
</tr>
</tbody>
</table>

$$\bar{x} = \frac{202.5\text{in.}^3}{40.5\text{in.}^2} = 5\text{in.}$$

$$\bar{y} = \frac{94.5\text{in.}^3}{40.5\text{in.}^2} = 2.33\text{in.}$$

Example 2 (pg 245)

Example Problem 7.3b (Figure 7.13)

An alternate method that can be employed in solving this problem is referred to as the negative area method.

A 6" thick concrete wall panel is precast to the dimensions as shown. Using the lower left corner as the reference origin, determine the center of gravity (centroid) of the panel.