Moments of Inertia

Notation:

- $A$ = name for area
- $b$ = name for a (base) width
- $d$ = calculus symbol for differentiation
  - name for a difference
  - name for a depth
- $d_x$ = difference in the x direction between an area centroid ($\tilde{x}$) and the centroid of the composite shape ($\hat{x}$)
- $d_y$ = difference in the y direction between an area centroid ($\tilde{y}$) and the centroid of the composite shape ($\hat{y}$)
- $h$ = name for a height
- $\bar{I}$ = moment of inertia about the centroid
- $I_c$ = moment of inertia of a component about the centroid
- $I_x$ = moment of inertia with respect to an x-axis
- $I_y$ = moment of inertia with respect to a y-axis
- $J_o$ = polar moment of inertia, as is $J$
- $O$ = name for reference origin
- $r_o$ = polar radius of gyration
- $r_x$ = radius of gyration with respect to an x-axis
- $r_y$ = radius of gyration with respect to a y-axis
- $t_f$ = thickness of a flange
- $t_w$ = thickness of web of wide flange
- $x$ = horizontal distance
- $\bar{x}$ = the distance in the x direction from a reference axis to the centroid of a shape
- $\hat{x}$ = the distance in the x direction from a reference axis to the centroid of a composite shape
- $y$ = vertical distance
- $\bar{y}$ = the distance in the y direction from a reference axis to the centroid of a shape
- $\hat{y}$ = the distance in the y direction from a reference axis to the centroid of a composite shape
- $t$ = thickness of a flange
- $P$ = plate symbol
- $\Sigma$ = summation symbol
- $\int$ = symbol for integration

- The cross section shape and how it resists bending and twisting is important to understanding beam and column behavior.

- **Definition:** Moment of Inertia; the second area moment

  \[ I_y = \int x^2 \, dA \quad I_x = \int y^2 \, dA \]

  We can define a single integral using a narrow strip:

  * for $I_x$, strip is parallel to $x$      for $I_y$, strip is parallel to $y$

  *I can be negative if the area is negative (a hole or subtraction).*

- A shape that has area at a greater distance away from an axis *through its centroid* will have a **larger** value of $I$. 

![](image1.png)
• Just like for center of gravity of an area, the moment of inertia can be determined with respect to any reference axis.

• **Definition:** **Polar Moment of Inertia:** the second area moment using polar coordinate axes

\[
J_o = \int r^2 dA = \int x^2 dA + \int y^2 dA
\]

\[
J_o = I_x + I_y
\]

• **Definition:** **Radius of Gyration:** the distance from the moment of inertia axis for an area at which the entire area could be considered as being concentrated at.

\[
I_x = r_x^2 A \Rightarrow r_x = \sqrt{\frac{I_x}{A}} \text{ radius of gyration in x}
\]

\[
r_y = \sqrt{\frac{I_y}{A}} \text{ radius of gyration in y}
\]

\[
r_o = \sqrt{\frac{J_o}{A}} \text{ polar radius of gyration, and } r_o^2 = r_x^2 + r_y^2
\]

**The Parallel-Axis Theorem**

• **The moment of inertia of an area with respect to any axis not through its centroid is equal to the moment of inertia of that area with respect to its own parallel centroidal axis plus the product of the area and the square of the distance between the two axes.**

\[
I = \int y^2 dA = \int (y'-d)^2 dA
\]

\[
= \int y'^2 dA + 2d \int y'dA + d^2 \int dA
\]

but \(\int y'dA = 0\), because the centroid is on this axis, resulting in:

\[
I_x = I_{cx} + Ad_y^2 \quad \text{(text notation) or } I_x = \bar{I}_x + Ad_y^2
\]

where \(I_{cx}\) (or \(\bar{I}_x\)) is the moment of inertia about the centroid of the area about an \(x\) axis and \(d_y\) is the \(y\) distance between the parallel axes
Similarly

\[ I_y = \bar{I}_y + Ad_s^2 \]

Moment of inertia about a \( y \) axis

\[ J_o = \bar{J}_o + Ad^2 \]

Polar moment of Inertia

\[ r_o^2 = \bar{r}_c^2 + d^2 \]

Polar radius of gyration

\[ r^2 = \bar{r}_c^2 + d^2 \]

Radius of gyration

* I can be negative again if the area is negative (a hole or subtraction).

** If \( \bar{I} \) is not given in a chart, but \( \bar{x} \) & \( \bar{y} \) are: YOU MUST CALCULATE \( \bar{I} \) WITH \( \bar{I} = I - Ad^2 \)

Composite Areas:

\[ I = \sum \bar{I} + \sum Ad^2 \]

where \( \bar{I} \) is the moment of inertia about the centroid of the component area

\( d \) is the distance from the centroid of the component area to the centroid of the composite area (ie. \( d_y = \hat{y} - \bar{y} \))

Basic Steps

1. Draw a reference origin.
2. Divide the area into basic shapes
3. Label the basic shapes (components)
4. Draw a table with headers of

   Component, Area, \( \bar{x} \), \( \bar{y} \), \( \bar{I}_x \), \( \bar{I}_y \), \( \bar{I}_x \), \( \bar{I}_y \), \( Ad_s^2 \), \( Ad_c^2 \)

5. Fill in the table values needed to calculate \( \hat{x} \) and \( \hat{y} \) for the composite
6. Fill in the rest of the table values.
7. Sum the moment of inertia (\( \bar{I} \)'s) and \( Ad^2 \) columns and add together.
### Geometric Properties of Areas

<table>
<thead>
<tr>
<th>Shape</th>
<th>Formula for Area</th>
<th>Central Axis Moment of Inertia</th>
<th>Sectional Axis Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rectangle</strong></td>
<td>Area = bh</td>
<td>$I_x = \frac{1}{12}bh^3$</td>
<td>$I_y = \frac{1}{12}bh^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_x = \frac{1}{12}bh^3$</td>
<td>$\bar{I}_y = \frac{1}{12}bh^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_c = \frac{1}{12}bh(b^2 + h^2)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_x = bh^3 / 12$</td>
<td>$\bar{I}_y = bh^3 / 12$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_c = bh(b^2 + h^2)$</td>
<td></td>
</tr>
<tr>
<td><strong>Triangle</strong></td>
<td>Area = $bh/2$</td>
<td>$\bar{I}_x = \frac{1}{6}bh^3$</td>
<td>$\bar{I}_y = \frac{1}{3}bh^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_c = \frac{1}{6}bh(b^2 + h^2)$</td>
<td></td>
</tr>
<tr>
<td><strong>Circle</strong></td>
<td>Area = $\pi r^2 = \pi d^2/4$</td>
<td>$I_x = \frac{1}{4} \pi r^4$</td>
<td>$I_y = \frac{1}{8} \pi r^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_x = 0.1098r^4$</td>
<td>$\bar{I}_y = 0.0549r^4$</td>
</tr>
<tr>
<td><strong>Semicircle</strong></td>
<td>Area = $\pi r^2/2 = \pi d^2/8$</td>
<td>$\bar{I}_x = 0.0549r^4$</td>
<td>$\bar{I}_y = 0.0549r^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_x = 0.0549r^4$</td>
<td>$\bar{I}_y = 0.0549r^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_x = 0.0549r^4$</td>
<td>$\bar{I}_y = 0.0549r^4$</td>
</tr>
<tr>
<td><strong>Quarter circle</strong></td>
<td>Area = $\pi r^2/4 = \pi d^2/16$</td>
<td>$\bar{I}_x = 0.0549r^4$</td>
<td>$\bar{I}_y = 0.0549r^4$</td>
</tr>
<tr>
<td><strong>Ellipse</strong></td>
<td>Area = $\pi ab$</td>
<td>$I_x = \frac{1}{4} \pi ab^3$</td>
<td>$I_y = \frac{1}{4} \pi ab^3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{I}_x = 16ah^3/175$</td>
<td>$\bar{I}_y = 4a^3h/15$</td>
</tr>
<tr>
<td><strong>Parabolic area</strong></td>
<td>Area = $4ah/3$</td>
<td>$\bar{I}_x = 37ah^3/2100$</td>
<td>$\bar{I}_y = a^3h/80$</td>
</tr>
<tr>
<td>**Parabolic span-</td>
<td>Area = $ah/3$</td>
<td>$\bar{I}_x = 3a^3h/4$</td>
<td>$\bar{I}_y = 3a^3h/10$</td>
</tr>
<tr>
<td>drel</td>
<td></td>
<td>$\bar{X} = 3a/4$</td>
<td>$\bar{Y} = 3h/10$</td>
</tr>
</tbody>
</table>

For the parabolic area and parabolic span-drel, the moments of inertia are calculated with respect to the centroidal axes. The expressions for the moments of inertia are given in terms of the specific coordinates and dimensions of each shape. Additionally, the central and sectional axis moments of inertia are also provided for reference, with the notation $\bar{I}$ indicating the moment of inertia about the centroidal axis.
Example 1 (pg 257)

Find the moments of inertia (\(\hat{x} = 3.05''\), \(\hat{y} = 1.05''\)).

<table>
<thead>
<tr>
<th>Component</th>
<th>(I_{xc}) (in.(^4))</th>
<th>(d_x) (in.)</th>
<th>(Ad_{x}^2) (in.(^6))</th>
<th>(I_{yc}) (in.(^4))</th>
<th>(d_y) (in.)</th>
<th>(Ad_{y}^2) (in.(^6))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{(1)(4)^3}{12})</td>
<td>5.33</td>
<td>0.95</td>
<td>3.61</td>
<td>(\frac{(4)(1)^3}{12})</td>
<td>0.33</td>
<td>2.55</td>
</tr>
<tr>
<td>(\frac{(7)(1)^3}{12})</td>
<td>0.58</td>
<td>0.55</td>
<td>2.12</td>
<td>(\frac{(1)(7)^3}{12})</td>
<td>28.58</td>
<td>1.45</td>
</tr>
<tr>
<td>(\sum I_{xc} = 5.91)</td>
<td>(\sum Ad_{x}^2 = 5.73)</td>
<td>(\sum I_{yc} = 28.91)</td>
<td>(\sum Ad_{y} = 40.73)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 2 (pg 253)

Example Problem 7.6 (Figures 7.24 to 7.26)

Determine the \(I\) about the centroidal x-axis.
Example 3
Determine the moments of inertia about the centroid of the shape.

Solution:

There is no reference origin suggested in figure (a), so the bottom left corner is good.

In figure (b) area A will be a complete rectangle, while areas C and A are "holes" with negative area and negative moment of inertias.

Area A = 200 mm x 100 mm = 20000 mm$^2$

$I_x = (200 mm)(100 mm)^3/12 = 16.667 \times 10^6 mm^4$

$I_y = (200 mm)^3(100 mm)/12 = 66.667 \times 10^6 mm^4$

Area B = $-\pi(30 mm)^2 = -2827.4 mm^2$

$I_x = I_y = -\pi (30 mm)^4/4 = -0.636 \times 10^6 mm^4$

Area C = $-1/2 \pi(50 mm)^2 = 3927.0 mm^2$

$I_x = -\pi (50 mm)^4/8 = -2.454 \times 10^6 mm^4$

$I_y = -0.1098(50 mm)^4 = -0.686 \times 10^6 mm^4$

Area D = 100 mm x 200 mm x 1/2 = 10000 mm$^2$

$I_x = (200 mm)(100 mm)^3/36 = 5.556 \times 10^6 mm^4$

$I_y = (200 mm)^3(100 mm)/36 = 22.222 \times 10^6 mm^4$

<table>
<thead>
<tr>
<th>shape</th>
<th>A (mm$^2$)</th>
<th>$\bar{x}$ (mm)</th>
<th>$\bar{y}$ (mm)</th>
<th>$\bar{x}A$ (mm$^3$)</th>
<th>$\bar{y}A$ (mm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>20000</td>
<td>100</td>
<td>50</td>
<td>2000000</td>
<td>1000000</td>
</tr>
<tr>
<td>B</td>
<td>-2827.43</td>
<td>150</td>
<td>-424115</td>
<td>-424115</td>
<td>-141372</td>
</tr>
<tr>
<td>C</td>
<td>-3926.99</td>
<td>21.22066</td>
<td>-83333.3</td>
<td>-83333.3</td>
<td>-196350</td>
</tr>
<tr>
<td>D</td>
<td>10000</td>
<td>66.6667</td>
<td>133.3333</td>
<td>666666.7</td>
<td>1333333</td>
</tr>
<tr>
<td></td>
<td>23245.58</td>
<td>2159218</td>
<td>1995612</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \hat{x} = \frac{2159218 mm^3}{23245.58 mm^2} = 92.9 \text{ mm} \]

\[ \hat{y} = \frac{1995612 mm^3}{23245.58 mm^2} = 85.8 \text{ mm} \]

<table>
<thead>
<tr>
<th>shape</th>
<th>$I_x$ (mm$^4$)</th>
<th>$d_y$ (mm)</th>
<th>$A_{d_x}^2$ (mm$^4$)</th>
<th>$I_y$ (mm$^4$)</th>
<th>$d_x$ (mm)</th>
<th>$A_{d_x}^2$ (mm$^4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>16666667</td>
<td>35.8</td>
<td>25632800</td>
<td>6666667</td>
<td>-7.1</td>
<td>1008200</td>
</tr>
<tr>
<td>B</td>
<td>-636173</td>
<td>35.8</td>
<td>-3623751.73</td>
<td>-636173</td>
<td>-57.1</td>
<td>-9218592.093</td>
</tr>
<tr>
<td>C</td>
<td>-2454369</td>
<td>35.8</td>
<td>-5032988.51</td>
<td>-686250</td>
<td>71.67934</td>
<td>-20176595.22</td>
</tr>
<tr>
<td>D</td>
<td>5555556</td>
<td>-47.5333</td>
<td>22594177.8</td>
<td>22222222</td>
<td>26.23333</td>
<td>6881876.029</td>
</tr>
<tr>
<td></td>
<td>19131680</td>
<td>39570237.5</td>
<td>87566466</td>
<td></td>
<td></td>
<td>-21505111.29</td>
</tr>
</tbody>
</table>

So,

\[ I_x = 19131680 + 39570237.5 = 58701918 = 58.7 \times 10^6 \text{ mm}^4 \]

\[ I_x = 87566466 + (-21505111.3 = 43572025 = 66.1 \times 10^6 \text{ mm}^4 \]
Example 4 (pg 258)
Example Problem 7.10 (Figures 7.35 and 7.36)

Locate the centroidal x and y axes for the cross-section shown. Use the reference origin indicated and assume that the steel plate is centered over the flange of the wide-flange section. Compute the $I_x$ and $I_y$ about the major centroidal axes.
Example 5 (pg 249)*

Example Problem 7.5 (Figures 7.16 and 7.17)

A composite or built-up cross-section for a beam is fabricated using two 1 1/2” × 10” vertical plates with a C12 × 20.7 channel section welded to the top and a W12 × 16 section welded to the bottom as shown. Determine the location of the major x-axis using the center of the W12 × 16’s web as the reference origin. Also determine the moment of inertia about both major centroidal axes.

<table>
<thead>
<tr>
<th>shape</th>
<th>A (in²)</th>
<th>(\bar{x}) (in)</th>
<th>(\bar{A}) (in³)</th>
<th>(\bar{y}) (in)</th>
<th>(\bar{A}) (in³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>channel</td>
<td>6.09</td>
<td>0</td>
<td>0.00</td>
<td>9.694</td>
<td>59.04</td>
</tr>
<tr>
<td>left plate</td>
<td>5</td>
<td>-3.25</td>
<td>-16.25</td>
<td>5.11</td>
<td>25.55</td>
</tr>
<tr>
<td>right plate</td>
<td>5</td>
<td>3.25</td>
<td>16.25</td>
<td>5.11</td>
<td>25.55</td>
</tr>
<tr>
<td>wide flange</td>
<td>4.71</td>
<td>0</td>
<td>0.00</td>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>20.80</td>
<td></td>
<td>0.00</td>
<td>110.14</td>
<td></td>
</tr>
</tbody>
</table>

\[ I_x = 0 \text{ in}^3 = 0 \text{ in} \]
\[ I_y = \frac{110.14 \text{ in}^3}{20.8 \text{ in}^2} = 5.295 \text{ in} \]

<table>
<thead>
<tr>
<th>shape</th>
<th>(l_x) (in⁴)</th>
<th>(d_y) (in)</th>
<th>(A_{d_y}^2) (in⁴)</th>
<th>(l_y) (in⁴)</th>
<th>(d_x) (in)</th>
<th>(A_{d_x}^2) (in⁴)</th>
</tr>
</thead>
<tbody>
<tr>
<td>channel</td>
<td>3.880</td>
<td>-4.399</td>
<td>117.849</td>
<td>129.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>left plate</td>
<td>41.667</td>
<td>0.185</td>
<td>0.171</td>
<td>0.104</td>
<td>3.250</td>
<td>52.813</td>
</tr>
<tr>
<td>right plate</td>
<td>41.667</td>
<td>0.185</td>
<td>0.171</td>
<td>0.104</td>
<td>-3.250</td>
<td>52.813</td>
</tr>
<tr>
<td>wide flange</td>
<td>2.800</td>
<td>5.295</td>
<td>132.054</td>
<td>103.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>90.013</td>
<td></td>
<td>250.245</td>
<td>232.208</td>
<td></td>
<td>105.625</td>
</tr>
</tbody>
</table>

\[ I_x = 90.013 + 250.245 = 340.259 = 340.3 \text{ in}^4 \]

\[ I_y = 232.208 + 105.625 = 337.833 = 337.8 \text{ in}^4 \]