Centers of Gravity - Centroids

- The center of gravity is the location of the equivalent force representing the total weight of a body comprised of particles that each have a mass gravity acts upon.

Resultant force: Over a body of constant thickness in \( x \) and \( y \)

\[
\sum F_z = \sum_{i=1}^{n} \Delta W_i = W \\
W = \int \mathrm{d}W
\]

Location: \( \bar{x}, \bar{y} \) is the equivalent location of the force \( W \) from all \( \Delta W_i \)'s over all \( x \) & \( y \) locations (with respect to the moment from each force) from:

\[
\sum M_y = \sum_{i=1}^{n} x_i \Delta W_i = \bar{x}W \\
\bar{x}W = \int x \mathrm{d}W \Rightarrow \bar{x} = \frac{\int x \mathrm{d}W}{W} \quad \text{OR} \quad \bar{x} = \frac{\sum (x \Delta W)}{W}
\]

\[
\sum M_x = \sum_{i=1}^{n} y_i \Delta W_i = \bar{y}W \\
\bar{y}W = \int y \mathrm{d}W \Rightarrow \bar{y} = \frac{\int y \mathrm{d}W}{W} \quad \text{OR} \quad \bar{y} = \frac{\sum (y \Delta W)}{W}
\]

- The centroid of an area is the average \( x \) and \( y \) locations of the area particles

For a discrete shape (\( \Delta A_i \)) of a uniform thickness and material, the weight can be defined as:

\[
\Delta W_i = \gamma t \Delta A_i \quad \text{where:} \quad \gamma \text{ is weight per unit thickness (specific weight) with units of } \text{N/m}^3 \text{ or lb/ft}^3 \\
t \Delta A_i \text{ is the volume}
\]

So if \( W = \gamma t A \):

\[
\bar{x} \gamma t A = \int x \gamma t \mathrm{d}A \Rightarrow \bar{x}A = \int x \mathrm{d}A \quad \text{OR} \quad \bar{x} = \frac{\sum (x \Delta A)}{A} \quad \text{and similarly} \quad \bar{y} = \frac{\sum (y \Delta A)}{A}
\]

Similarly, for a line with constant cross section, \( a \) (\( \Delta W_i = \gamma a \Delta L_i \)):

\[
\bar{x}L = \int x \mathrm{d}L \quad \text{OR} \quad \bar{x} = \frac{\sum (x \Delta L)}{L} \quad \text{and} \quad \bar{y}L = \int y \mathrm{d}L \quad \text{OR} \quad \bar{y} = \frac{\sum (y \Delta L)}{L}
\]

- \( \bar{x}, \bar{y} \) with respect to an \( x, y \) coordinate system is the centroid of an area AND the center of gravity for a body of uniform material and thickness.
- The first moment of the area is like a force moment: and is the \( \bar{y} \) multiplied by the perpendicular distance to an axis.

\[
Q_x = \int y \, dA = \bar{y} \, A \\
Q_y = \int x \, dA = \bar{x} \, A
\]

- Centroids of Common Shapes

<table>
<thead>
<tr>
<th>Shape</th>
<th>( \bar{x} )</th>
<th>( \bar{y} )</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular area</td>
<td>( \frac{b}{3} )</td>
<td>( \frac{h}{3} )</td>
<td>( \frac{bh}{2} )</td>
</tr>
<tr>
<td>Quarter-circular area</td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{\pi r^2}{4} )</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{\pi r^2}{2} )</td>
</tr>
<tr>
<td>Semiparabolic area</td>
<td>( \frac{3a}{8} )</td>
<td>( \frac{3h}{5} )</td>
<td>( \frac{2ah}{3} )</td>
</tr>
<tr>
<td>Parabolic area</td>
<td>0</td>
<td>( \frac{3h}{5} )</td>
<td>( \frac{4ah}{3} )</td>
</tr>
<tr>
<td>Parabolic span-drel</td>
<td>( \frac{3a}{4} )</td>
<td>( \frac{3h}{10} )</td>
<td>( \frac{ah}{3} )</td>
</tr>
<tr>
<td>Circular sector</td>
<td>( \frac{2r \sin \alpha}{3\alpha} )</td>
<td>0</td>
<td>( \alpha r^2 )</td>
</tr>
<tr>
<td>Quarter-circular arc</td>
<td>( \frac{2r}{\pi} )</td>
<td>( \frac{2r}{\pi} )</td>
<td>( \frac{\pi r}{2} )</td>
</tr>
<tr>
<td>Semicircular arc</td>
<td>0</td>
<td>( \frac{2r}{\pi} )</td>
<td>( \pi r )</td>
</tr>
<tr>
<td>Arc of circle</td>
<td>( \frac{r \sin \alpha}{\alpha} )</td>
<td>0</td>
<td>2( \alpha r )</td>
</tr>
</tbody>
</table>
• **Symmetric Areas**
  - An area is symmetric with respect to a line when every point on one side is mirrored on the other. The line divides the area into equal parts and the centroid will be on that axis.
  - An area can be symmetric to a center point when every (x,y) point is matched by a (-x,-y) point. It does not necessarily have an axis of symmetry. The center point is the centroid.
  - If the symmetry line is on an axis, the centroid location is on that axis (value of 0). With double symmetry, the centroid is at the intersection.
  - Symmetry can also be defined by areas that match across a line, but are 180° to each other.

**Basic Steps**

1. Draw a reference origin.
2. Divide the area into basic shapes
3. Label the basic shapes (components)
4. Draw a table with headers of Component, Area, \( \bar{x} \), \( \bar{y} \), \( \bar{A} \)
5. Fill in the table value
6. Draw a summation line. Sum all the areas, all the \( \bar{x}A \) terms, and all the \( \bar{y}A \) terms
7. Calculate \( \bar{x} \) and \( \bar{y} \)

• **Composite Shapes**
  
  If we have a shape made up of basic shapes that we know centroid locations for, we can find an “average” centroid of the areas.

\[
\bar{x}A = \bar{x} \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \bar{x}_i A_i \\
\bar{y}A = \bar{y} \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \bar{y}_i A_i
\]

*Centroid values can be negative.*
*Area values can be negative (holes)*
Example 1 (pg 243)
Example Problem 7.1: Centroids (Figures 7.5 and 7.6)
Determine the centroidal x and y distances for the composite area shown. Use the lower left corner of the trapezoid as the reference origin.

<table>
<thead>
<tr>
<th>Component</th>
<th>Area (ΔA) (in.²)</th>
<th>x (in.)</th>
<th>xΔA (in.³)</th>
<th>y (in.)</th>
<th>yΔA (in.³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(9\text{'}(3\text{'}) \div 2 = 13.5 \text{ in.}²)</td>
<td>6\text{'}</td>
<td>81 \text{ in.}³</td>
<td>4\text{'}</td>
<td>54 \text{ in.}³</td>
</tr>
<tr>
<td>(b)</td>
<td>9\text{'} (3\text{'}) \div 2 = 27 \text{ in.}²</td>
<td>4.5\text{'}</td>
<td>121.5 \text{ in.}³</td>
<td>1.5\text{'}</td>
<td>40.5 \text{ in.}³</td>
</tr>
</tbody>
</table>

\[
\bar{x} = \frac{\sum x \Delta A}{\sum \Delta A} = \frac{202.5 \text{ in.}³}{40.5 \text{ in.}²} = 5 \text{ in.}
\]

\[
\bar{y} = \frac{\sum y \Delta A}{\sum \Delta A} = \frac{94.5 \text{ in.}³}{40.5 \text{ in.}²} = 2.33 \text{ in.}
\]

Example 2 (pg 245)
Example Problem 7.3b (Figure 7.13)
An alternate method that can be employed in solving this problem is referred to as the negative area method.

A 6\text{'} thick concrete wall panel is precast to the dimensions as shown. Using the lower left corner as the reference origin, determine the center of gravity (centroid) of the panel.
Example 3 (pg 249)
Example Problem 7.5 (Figures 7.16 and 7.17)

A composite or built-up cross-section for a beam is fabricated using two $\frac{1}{2}'' \times 10''$ vertical plates with a C12 x 20.7 channel section welded to the top and a W12 x 16 section welded to the bottom as shown. Determine the location of the major x-axis using the center of the W12 x 16’s web as the reference origin.