Examples:
Rigid Frames

Example 1  From eStructures v1.1, Schodek and Pollalis, 2000 Harvard College

Rigid Frame Structures: Lateral Loading
Pinned Base Connections

Assumed directions of reactions:
Horizontal components balance applied force
Vertical components act as shown to prevent overturning

ΣM_A = 0
+ 2000(12) - R_C(30) = 0
R_C = 800

ΣF_y = 0
+ R_A - 800 = 0 or R_A = 800

ΣF_x = 0
R_A_x + R_C_x = 2000

This last equation cannot be solved by statics alone. The structure is actually statically indeterminate. As shown on the following slides, an approximate method of analysis can be used to find the unknown reactions.
Example 1 (continued)

RIGID FRAME STRUCTURES

DRAW DEFLECTED SHAPE OF STRUCTURE

Each of the top rigid joints translates and rotates as a unit.

A "point of inflection" naturally develops at the midspan of the horizontal member. This is a point of reverse curvature in the member, and hence is a point of zero moment.

If use is made of the point of inflection as a point of known zero moment, the structure is now statically determinate and can analyzed much like a three-hinged arch.

RIGID FRAME STRUCTURES

ANALYZE RIGHT PART

The forces shown at B are internal to the structure.
Example 1 (continued)

FOR RIGHT PART:
\[ \sum M_B = 0 \]
\[ -R_{C_y}(15) + R_{C_y}(12) = 0 \]
\[ \text{or} \quad R_{C_y} = 1000 \]

FOR RIGHT PART:
\[ \sum F_y = 0 \]
\[ R_{B_y} - R_{C_y} = 0 \quad \text{or} \quad R_{B_y} = 800 \]
\[ \sum F_x = 0 \]
\[ R_{B_x} + R_{C_x} = 2000 \quad \text{or} \quad R_{B_x} = 1000 \]
Example 1 (continued)
Example 1 (continued)

RIGID FRAME STRUCTURES

Point of Inflection

$P = 2000 \text{ lb}$

$R_{Ax}$  $R_{Ay}$  $R_{Cx}$  $R_{Cy}$  $R_{Ax}$  $R_{Ay}$  $R_{Cx}$  $R_{Cy}$

30 ft

12 ft

15 ft

15 ft

$P = 2000 \text{ lb}$

REATIONS AT A & C AND THE INTERNAL FORCES AT B

RIGID FRAME STRUCTURES

Point of Inflection

$P = 2000 \text{ lb}$

$R_{Bx}$  $R_{By}$

12 ft

12 ft

15 ft

$R_{Bx} = 1000$

$R_{By} = 800$

ANALYZE LEFT PART

$R_{Ax} = 1000$

$R_{Cy} = 800$

FOR LEFT PART:

$\Sigma F_y = 0 \quad R_{Ay} = 1000$

$\Sigma F_x = 0 \quad R_{Cy} = 1000$

$\Sigma M_B = 0 \quad -800(15) + 1000(12) = 0$

Check!
Example 1 (continued)

RIGID FRAME STRUCTURES

P = 2000 lb

ANALYZE LEFT PART

15 ft

R_B = 1000

R_A = 1000

R_C = 800

R_A = 800

R_C = 800

BENDING MOMENTS

M_ED = -800(15) = -12000 ft-lb

M_ED = -1000(12) = -12000 ft-lb

V_E = 800

V_A = -1000

SHEAR

RIGID FRAME STRUCTURES: FINAL RESULTS

P = 2000 lb

M = -1200 ft lb

M = 1200 ft lb

V = 800

V = -1000

FINAL BENDING MOMENTS

FINAL SHEAR

V = -1000

V = -1000

FINAL REACTIONS AT A & C

AND THE INTERNAL FORCES

AT B

R_B = 1281 lb

R_B = 1281 lb

R_A = 1281 lb

R_C = 1281 lb
**Example 2**

The rigid frame shown at the right has the loading and supports as show. Using superpositioning from approximate analysis methods, draw the shear and bending moment diagrams.

**Solution:**

*Reactions* The two loading situations for which approximate reaction values are available are shown below. These values must be calculated and added together (allowed by superpositioning).

\[
\begin{align*}
R_{AH} &= 0.907wh \\
R_{AV} &= 0.197wh^2/L \\
MR_A &= 0.303wh^2 \\
R_{DH} &= 0.093wh \\
R_{DV} &= 0.197wh^2/L \\
R_{DH} &= -0.0551Ph/L \\
R_{DV} &= 0.0551Ph/L \\
R_{DV} &= 0.516P
\end{align*}
\]

\[
\begin{align*}
R_{AH} &= -0.907wh + 0.0551Ph/L = -0.907(10^{kN/m})(6m) + \frac{0.0551(50kN)(6m)}{5m} = -51.11 \text{ kN} \\
R_{AV} &= -0.197wh^2/L + 0.484P = -0.197(10^{kN/m})(6m)^2 + 0.484(50kN) = 10.02 \text{ kN} \\
MR_A &= -0.303wh^2 + 0.0112Ph^2/L = -0.303(10^{kN/m})(6m)^2 + \frac{0.0112(50kN)(6m)^2}{5m} = -105.05 \text{ kN-m} \\
R_{DH} &= -0.093wh - 0.0551Ph/L = -0.093(10^{kN/m})(6m) - \frac{0.0551(50kN)(6m)}{5m} = -8.89 \text{ kN} \\
R_{DV} &= 0.197wh^2/L + 0.516P = \frac{0.197(10^{kN/m})(6m)^2}{5m} + 0.516(50kN) = 39.98 \text{ kN}
\end{align*}
\]

*Member End Forces* The free-body diagrams of all the members and joints of the frame are shown below. The unknowns on the members are drawn as anticipated, and the opposite directions are drawn on the joint. We can begin the computation of internal forces with either member AB or CD, both of which have only three unknowns.
**Member AB**  With the magnitudes of reaction forces at A known, the unknowns are at end B of BA\textsubscript{x}, BA\textsubscript{y}, and M\textsubscript{BA}, which can be determined by applying \( \Sigma F_x = 0 \), \( \Sigma F_y = 0 \), and \( \Sigma M_B = 0 \). (NOTE: The reaction directions are assumed. When the results are positive, the assumption is verified.) Thus,

\[
\Sigma F_x = -51.11 \text{kN} + 10 \text{kN/(6m) - BA}_x = 0 \quad \text{BA}_x = 8.89 \text{kN}, \quad \Sigma F_y = 10.02 \text{kN} - BA_y = 0 \quad \text{BA}_y = 10.02 \text{kN}
\]

\[
\Sigma M_A = 105.05 \text{kN-m} - 10 \text{kN/m}(6m)(3m) + 8.89 \text{kN}(6m) + M_{BA} = 0 \quad M_{BA} = 21.16 \text{kN-m}
\]

**Joint B**  Because the forces and moments must be equal and opposite, BC\textsubscript{x} = 8.89 kN, BC\textsubscript{y} = 10.02 kN and M\textsubscript{BC} = 21.16 kN-m. (NOTE: The directions are opposite on the other side of the section cut.)

**Member CD**  With the magnitudes of reaction forces at D known, the unknowns are at end C of CD\textsubscript{x}, CD\textsubscript{y}, and M\textsubscript{CD}, which can be determined by applying \( \Sigma F_x = 0 \), \( \Sigma F_y = 0 \), and \( \Sigma M_B = 0 \). Thus,

\[
\Sigma F_x = -8.89 \text{kN} - CD_x = 0 \quad CD_x = -8.89 \text{kN (opposite arrow direction!)}
\]

\[
\Sigma F_y = 39.98 \text{kN} - CD_y = 0 \quad CD_y = 39.98 \text{kN}
\]

\[
\Sigma M_D = -8.89 \text{kN}(6m) + M_{CD} = 0 \quad M_{DC} = 53.34 \text{kN-m}
\]

**Joint C**  Because the forces and moments must be equal and opposite, CB\textsubscript{x} = -8.89 kN (again, the actual direction is opposite what is shown on the FBD), CB\textsubscript{y} = 39.98 kN and M\textsubscript{CB} = 53.34 kN-m

**Member BC**  All forces are known, so equilibrium can be checked.

\[
\Sigma F_x = 8.89 \text{kN} + (-8.89 \text{kN}) = 0 \quad (CB_y is shown to the right in the FBD but has a negative value.)
\]

\[
\Sigma F_y = 10.02 \text{kN} - 50 \text{kN} + 39.98 \text{kN} = 0
\]

\[
\Sigma M_B = -21.16 \text{kN-m} - 50 \text{kN}(2.5m) - 53.34 \text{kN-m} + 39.98 \text{kN}(5m) = 0.4 = 0
\]

(Remember: To find the point of zero shear with a distributed load, divide the peak \{triangle\} shear by the distributed load; ex. 51.11kN/(10 kN/m) = 5.11 m)
Example 3

Using Multiframe, verify the bending moment diagram for the example in Figure 9.9:

Assuming steel \((E = 29,000 \text{ ksi})\)

### Joint Coordinates (ft)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Label</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
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### Section Properties

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Figure 9.9 The moment distribution illustrates the importance of relative stiffness considerations. The values obtained are quite different from those obtained by estimating points of inflection and using hand calculations.
Example 3 (continued)

Displacement:

Maximum Actions for all members (column-1, beam-2, column-3):

<table>
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<tr>
<th>Memb</th>
<th>Label</th>
<th>Section</th>
<th>Sign</th>
<th>Px' kip</th>
<th>Vy' kip</th>
<th>Vz' kip</th>
<th>Tx' kip-ft</th>
<th>My' kip-ft</th>
<th>Mz' kip-ft</th>
<th>dy' in</th>
<th>dz' in</th>
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<td>109.079</td>
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(axes orientation reference)

Maximum Stresses for all members (column-1, beam-2, column-3):

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<th>Label</th>
<th>Section</th>
<th>Sign</th>
<th>Sb' top ksi</th>
<th>Sb' bot ksi</th>
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<th>Sx'+Sb' bot kisi</th>
<th>dy' in</th>
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Beam-Column stress verification (combined stresses) when d = 24 in, A = 28 in², Iₙ = 2380 in⁴:

\[
f_{\text{max}} = \frac{P}{A} + \frac{M}{S} = \frac{P}{A} + \frac{Mc}{I} = \frac{216k}{28in^2} + \frac{1425^{1/2} \cdot (24in^2/2)}{2380in^4} \cdot \frac{12in}{ft} = 7.71ksi + 86.22ksi = 93.93ksi
\]